DOUBLE WAVES IN AN IDEAL RIGIDLY PLASTIC BODY UNDER PLANE DEFORMATION

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One method to obtain exact solutions of systems of partial differential equations is the method of the degenerate hodograph. Extensive application of solutions with a degenerate hodograph in gasdynamics [1] permits the hope for successful application of this method in plasticity theory. The crux of the method is to reduce the dimensionality of the independent variables by imposing finite relations between the dependent variables. Solutions obtained by such a method are partially invariant [2] from the viewpoint of a group classification. Simple waves of systems with two independent variables [3, 4] were used from the solutions with degenerate hodograph in plasticity theory. When the number of independent variables is greater than two, individual examples are known for the construction of simple [5] and double [6] waves in plasticity theory. An attempt is made in [6] to approach solutions with a degenerate hodograph from the viewpoint of generalized traveling waves and Riemann invariants when the number of independent variables is greater than two. This resulted in a limiting condition on the existence of solutions of the form of double waves (in the sense of [6]). In this paper a complete classification is given of double waves with functional arbitrariness for the equations of motion of an ideal rigidly plastic body under plane strain

$$\rho\frac{\partial v_i}{\partial t}=\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha},\quad \frac{\partial v_\alpha}{\partial x_\alpha}=0\qquad (i=1,2),$$
$$\frac{\sigma_{11}-\sigma_{22}}{2\sigma_{12}}=\frac{\partial v_1/\partial x_1-\partial v_2/\partial x_2}{\partial v_1/\partial x_2+\partial v_2/\partial x_1},\quad (\sigma_{11}-\sigma_{22})^2+4\sigma_{12}^2=4k^2. \tag{0.1}$$

The notation here is customary. Summation from 1 to 2 is performed over repeated Greek subscripts. Without limiting the generality it can be considered that $\rho = 1$.

After substitution into (0.1), the relationships $\sigma_{11} = \sigma - k\sin 2\theta$, $\sigma_{22} = \sigma + k\sin 2\theta$, $\sigma_{12} = k\cos 2\theta$, a system of four differential equations will be obtained for $\sigma(t, x_1, x_2)$, $\theta(t, x_1, x_2)$, $v_i(t, x_1, x_2)$ $(i = 1, 2)$

$$\partial v_1/\partial t=\partial\sigma/\partial x_1-2k\,(\cos 2\theta\,\partial\theta/\partial x_1+\sin 2\theta\,\partial\theta/\partial x_2),$$
$$\partial v_2/\partial t=\partial\sigma/\partial x_2-2k(\sin 2\theta\,\partial\theta/\partial x_1-\cos 2\theta\,\partial\theta/\partial x_2),$$
$$\partial v_\alpha/\partial x_\alpha=0,\ \partial v_2/\partial x_1+\partial v_1/\partial x_2-2\,\mathrm{ctg}\,2\theta\,\partial v_2/\partial x_2=0. \tag{0.2}$$

The trivial case θ = const is excluded from further consideration.

1. Let the velocities v_1 and v_2 be functionally independent in a double wave. Then the variables v_1 and v_2 can be taken as the double wave parameters, i.e., we can set

$$\sigma=\sigma(v_1,v_2),\ \theta=\theta(v_1,v_2). \tag{1.1}$$

After substituting (1.1) into (0.2), an overdefined system of four differential equations is obtained in the two functions v_1, v_2 $(x \equiv x_1,\ x_2 \equiv y)$

$$\mathbf{v}_t+G_1\mathbf{v}_y=0,\ \mathbf{v}_x+G_2\mathbf{v}_y=0, \tag{1.2}$$

where $\mathbf{v} = (v_1, v_2)'$ and the matrices G_1 and G_2 have the form

$$G_1=\begin{pmatrix} a_1 & a_2-a_1\,2\,\mathrm{ctg}\,2\theta\\ -a_2 & -a_1\end{pmatrix},\quad G_2=\begin{pmatrix}0 & 1\\ 1 & -2\,\mathrm{ctg}\,2\theta\end{pmatrix},$$
$$a_1=\sigma_2+2k(\theta_1\sin 2\theta-\theta_2\cos 2\theta),$$

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$$a_2 = \sigma_1 + 2k(\theta_1 \cos 2\theta + \theta_2 \sin 2\theta),$$
$$\sigma_i = \partial\sigma/\partial v_i,\ \theta_i = \partial\theta/\partial v_i\ (i = 1,\ 2).$$

The greatest possible arbitrariness in the solution of the system (1.2) for given functions (1.1) is determined by the number $2 - r$ ($r = \mathrm{rank}(G_1G_2 - G_2G_1)$), as follows from an examination of the system obtained by continuing (1.2). Consequently, for the existence of the double waves of (0.1) with given functions (1.1) having functional arbitrariness in the solution, it is necessary that $r \leq 1$.

If $r = 0$, then $a_i = 0$ ($i = 1, 2$), which corresponds to stationary flow, i.e., a reduction occurs partially of the invariant solution to the invariant [2] (this is excluded from the subsequent consideration).

Remark. A corollary of the existence of double waves in the sense of [6] is the requirement $r = 0$.

Let us examine the case when $r = 1$. From this condition and from the form of the matrix $(G_1G_2 - G_2G_1)$

$$a_2 = a_1(\alpha + \cos 2\theta)/\sin 2\theta\ (\alpha = \pm 1). \tag{1.3}$$

If the Jacobian $\partial(v_1, v_2)/\partial(x, y) = 0$, then after going over from the independent variables (t, x, y) to new variables (v_1, v_2, x) or (v_1, v_2, y), the contradiction to functional independence of v_1 and v_2 is obtained. This means $\partial(v_1, v_2)/\partial(x, y) \neq 0$. Let us make the passage over to new independent variables (v_1, v_2, t)

$$x = P(v_1,\ v_2,\ t),\ y = Q(v_1,\ v_2,\ t) \tag{1.4}$$

in the system (1.2):

$$\begin{gathered} P_2Q_t - P_tQ_2 - a_1P_2 + P_1(a_2 - a_1 2 \operatorname{ctg} 2\theta) = 0, \\ P_1Q_t - P_tQ_1 - a_2P_2 + a_1P_1 = 0, \\ Q_2 + P_1 = 0,\ Q_1 + P_2 + 2 \operatorname{ctg} 2\theta P_1 = 0, \end{gathered} \tag{1.5}$$

where

$$\begin{gathered} P_i = \partial P/\partial v_i;\ Q_i = \partial Q/\partial v_i;\ P_t = \partial P/\partial t;\ Q_t = \partial Q/\partial t; \\ P_1Q_2 - P_2Q_1 \neq 0. \end{gathered} \tag{1.6}$$

Since (1.5) is linear and homogeneous as a system of algebraic equations in the variables P_i, Q_i ($i = 1, 2$), then by virtue of (1.6) its determinant must satisfy the equality

$$\Delta \equiv (P_t)^2 - (Q_t)^2 - 2P_tQ_t \operatorname{ctg} 2\theta - ((a_2)^2 - (a_1)^2 - 2a_1a_2 \operatorname{ctg} 2\theta) = 0.$$

Since a_1 and a_2 are connected by the relationship (1.3), then from the last equality

$$P_t = \lambda Q_t \tag{1.7}$$
$$(\lambda = (\beta + \cos 2\theta)/\sin 2\theta,\ \beta = \pm 1,\ \lambda_i = \partial\lambda/\partial v_i\ (i = 1,\ 2)).$$

Upon integrating (1.7) with respect to t there is obtained

$$P = \lambda Q + \chi(v_1,\ v_2). \tag{1.8}$$

After substitution of (1.8) into the last two equations of the system (1.5) we have

$$\begin{gathered} Q_2 + \lambda Q_1 + (\lambda_1 Q + \chi_1) = 0, \\ (\lambda - 2 \operatorname{ctg} 2\theta)Q_2 + Q_1 + (\lambda_2 Q + \chi_2) = 0. \end{gathered} \tag{1.9}$$

Since λ satisfies the equation $\lambda^2 - 2\lambda \cot 2\theta - 1 = 0$, then $(\lambda_1 - \lambda\lambda_2)Q + \chi_1 - \lambda\chi_2 = 0$ from (1.9). In the last relationship $Q_t = 0$ and $P_t = 0$ follow for $\lambda_1 - \lambda\lambda_2 \neq 0$ (this case is excluded from examination). Hence

$$\lambda_1 - \lambda\lambda_2 = 0,\ \chi_1 - \lambda\chi_2 = 0. \tag{1.10}$$

After substitution of (1.8) in the first two equations of the system (1.5) and taking (1.10) into account, $\alpha = \beta$ results and

$$(a_2\lambda + a_1)Q_1 + (a_2\lambda + Q_t)(\lambda_2 Q + \chi_2) = 0. \tag{1.11}$$

From the equality $\alpha = \beta$, (1.10), and the form of the functions λ and a_i (i = 1, 2), there follows $\sigma = \sigma(\lambda)$, $\chi = \chi(\lambda)$. Therefore, the function $Q(v_1, v_2, t)$ satisfies the system of two quasilinear differential equations: the first equation in (1.9) and (1.11). This system results in a linear homogeneous system whose compatibility is investigated by Poisson brackets, by a standard method. After obtaining one Poisson bracket, we have

$$Q_t(cQ + d) = 0,$$

where $c = (\partial/\partial v_1)((1 + \lambda^2)\sigma' - 2\beta k)^{-1}$, $d = (\partial/\partial v_1)(\chi'/((1 + \lambda^2)\sigma' - 2\beta k))$. Since $Q_t \neq 0$, then $c = 0$, $d = 0$ or after integration $(1 + \lambda^2)\sigma' - 2\beta k = \varphi(v_2)$, $\chi' = \psi(v_2)$ with arbitrary functions $\varphi(v_2)$, $\psi(v_2)$.

If $(d(\sigma'(1 + \lambda^2))/d\lambda)^2 + (\chi'')^2 \neq 0$, then $\lambda = \lambda(v_2)$, but then there follows from (1.10) that λ = const. Consequently

$$\chi = c_2\lambda - c_1; \tag{1.12}$$

$$\sigma + 2\beta k\theta + c_3\theta = c_4 \tag{1.13}$$

(c_i are constants, i = 1, 2, 3, 4).

After substitution of (1.12) into (1.4), by virtue of (1.8)

$$x + c_1 = \lambda(Q + c_2),\ y + c_2 = Q + c_2.$$

We have from the last equations

$$\lambda = (x + c_1)/(y + c_2). \tag{1.14}$$

There is thereby obtained from (1.13), (1.14) and $\lambda = \lambda(\theta)$ that the stress field in the double wave under consideration is "statically" determinable: the functions $\theta = \theta(x, y)$ and $\sigma = \sigma(x, y)$ are found from (1.13) and the relationships

$$\tan\theta = -(x + c_1)/(y + c_2),\ \beta = -1,\ \mathrm{tg}\,\theta = (y + c_2)/(x + c_1),\ \beta = 1.$$

Then follows from the "static" determinability of the stress field

$$v_i = tH_i(x, y) + g_i(x, y)\ (i = 1, 2), \tag{1.15}$$

where $H_i = (c_3\lambda_{x_i})/(1 + \lambda^2)$ (i = 1, 2) by virtue of the equations (0.2), (1.13), and (1.14) and the functions $g_i(x, y)$ satisfy the system of differential equations

$$\partial g_1/\partial x + \partial g_2/\partial y = 0,\ \partial g_2/\partial x + \partial g_1/\partial y + ((1 - \lambda^2)/\lambda)\partial g_2/\partial y = 0. \tag{1.16}$$

There is a general solution for $\lambda(x, y)$ from (1.14) for the system (1.16)

$$\begin{aligned} g_1 &= (\Phi_1 + \Phi_2 + \lambda(\lambda^2 + 1)\Phi_1')/\sqrt{1 + \lambda^2}, \\ g_2 &= ((1 + \lambda^2)\Phi_1' - \lambda(\Phi_1 + \Phi_2))/\sqrt{1 + \lambda^2} \end{aligned} \tag{1.17}$$

with arbitrary functions $\Phi_1 = \Phi_1(\lambda)$, $\Phi_2 = \Phi_2(r)$ of their arguments ($r = \sqrt{(x + c_1)^2 + (y + c_2)^2}$). In a polar (r, φ) ($x + c_1 = r\cos\varphi$, $y + c_2 = r\sin\varphi$) coordinate system the velocity field (1.15) and (1.17) obtained is written as

$$v_r = -\Phi_1',\ v_\varphi = c_3 t/r + \Phi_1 + \Phi_2.$$

Remark. The solution of (0.1) in which the velocity field has the form (1.15) while the stress field is independent of the time t is invariant relative to the subgroup generated by the infinitesimal operator [5] $N = t\partial_t + v_\alpha\partial_{x_\alpha}$.

2. Let us consider the case when the velocity components v_1 and v_2 are functionally dependent: $v_2 = \Phi(v_1)$.

After substituting $v_2 = \Phi(v_1)$ into the last two equations of the system (0.2), we obtain by virtue of the inequality $(\partial v_1/\partial x_1)^2 + (\partial v_2/\partial x_2)^2 \neq 0$ that $\theta = \theta(v_1)$, where

$$\Phi' = -(\alpha + \cos 2\theta)/\sin 2\theta \ (\alpha = \pm 1). \tag{2.1}$$

Here the variables (σ, v_1) can be selected as independent parameters in the double wave. Substituting $v_2 = \Phi(v_1)$ into the first two equations of the system (0.2), subtracting from the first equation (differentiated with respect to y), the second equation (differentiated with respect to x), using (2.1) and integrating with respect to t, we obtain

$$\partial v_1/\partial y - \Phi' \partial v_1/\partial x = g(x, y).$$

Then we deduce from the continuity equation and the last equation

$$\partial v_1/\partial y = g/(1 + (\Phi')^2),\ \partial v_1/\partial x = -g\Phi'/(1 + (\Phi')^2). \tag{2.2}$$

After cross differentiation of (2.2)

$$g_x + \Phi' g_y + \Phi'' g^2/(1 + (\Phi')^2) = 0.$$

If the last equation can be solved for v_1, then the double wave is reduced to the stationary solution. Therefore, there necessarily results

$$\Phi''/(1 + (\Phi')^2) = c_2\Phi' + c_1,\ g = (c_3 + c_1 x + c_2 y)^{-1} \tag{2.3}$$

(c_i are constants, i = 1, 2, 3). Without limiting the generality, it can be considered that $c_2 = 0$ (this is achieved by rotating the coordinate system). Upon integrating (2.2) with (2.3) being used, we obtain

$$\Phi'(v_1) = (y + h(t))/(x + c),\ (c = c_3/c_1) \tag{2.4}$$

with the arbitrary function $h = h(t)$. Taking account of (2.1), (2.3), and (2.4), we have from the motion equations

$$\sigma = 2\alpha k\theta + (h'/2c_1)\ln((x + c)^2 + (y + h)^2) + \mu(t). \tag{2.5}$$

Therefore, in the case of a functional dependence between v_1 and v_2 a solution in the form of a double wave is determined by (2.1), (2.3)-(2.5).

3. Solutions of the system (0.1) invariant relative to the infinitesimal operator N [5] should be sought in the form

$$\sigma = \sigma(x, y),\ \theta = \theta(x, y),\ v_i = tH_i(x, y) + g_i(x, y)\ (i = 1, 2). \tag{3.1}$$

After substituting the relationships (3.1) and (0.2) into the functions σ, θ, H_1, H_2 a closed system of differential equations is obtained

$$\begin{gathered} \partial R_1/\partial x + \operatorname{tg}\theta \partial R_1/\partial y = H_1 + H_2 \operatorname{tg}\theta, \\ \partial R_2/\partial x - \operatorname{ctg}\theta \partial R_2/\partial y = H_1 - H_2 \operatorname{ctg}\theta, \\ \partial H_1/\partial x + \partial H_2/\partial y = 0,\ \partial H_2/\partial x + \partial H_1/\partial y - 2\operatorname{ctg} 2\theta \partial H_2/\partial y = 0, \end{gathered} \tag{3.2}$$

and the functions $g_i(x, y)$ (i = 1, 2) are found by solving the system

$$\partial g_1/\partial x + \partial g_2/\partial y = 0,\ \partial g_2/\partial x + \partial g_1/\partial y - 2\operatorname{ctg} 2\theta \partial g_2/\partial y = 0$$

with the values $\theta(x, y)$ from the solution of the system (3.2) substituted therein. Here $R_i = \sigma + (-1)^i 2k\theta$ (i = 1, 2).

The basis of the Lie algebra that corresponds to a transformation group allowable by the system (3.2), is formed by the operators [2]

$$\begin{aligned} &X_i = \partial_{x_i},\quad X_3 = \partial_{R_1} + \partial_{R_2},\quad X_4 = H_\alpha \partial_{H_\alpha} - x_\alpha \partial_{x_\alpha}, \\ &X_5 = H_2\partial_{H_1} - H_1\partial_{H_2} + x_2\partial_{x_1} - x_1\partial_{x_2} + 2k(\partial_{R_1} - \partial_{R_2}), \\ &X_{5+i} = x_i(\partial_{R_1} + \partial_{R_2}) + \partial_{H_i} \quad (i = 1, 2). \end{aligned} \tag{3.3}$$

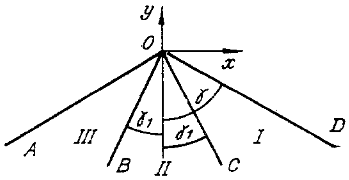

Fig. 1

The optimal system of one-parameter subalgebras for the Lie algebra (3.3) has the form

$$X_3,\ X_6,\ X_4+\gamma X_5+\beta X_3,$$
$$X_5+\beta X_3,\ X_1+\beta X_6+\gamma X_7$$

(γ, β are arbitrary constants; dissimilar subalgebras correspond to different values of γ and β).

The solution invariant relative to the subgroup generated by the operator X_4 is needed for later. It is written [in the polar (r, φ) coordinate system] as

$$\begin{gathered}H_\varphi=c_3/r,\ H_r=(-2k\sin 2(\theta-\varphi)+c_1\sin(c_2-2\varphi))/r,\\ 2k\cos 2(\theta-\varphi)=c_3+c_1\cos(c_2-2\varphi),\\ \sigma=(c_1\sin(c_2-2\varphi)-2k\sin 2(\theta-\varphi))/2+c_4.\end{gathered}\tag{3.4}$$

4. Let us consider the problem of a rigidly plastic state of plane strain for a wedge with aperture angle $2\gamma>\pi/2$ loaded by uniform pressure p on one face. In a static formulation this problem was considered by many authors (a sufficiently complete survey is in [7]). The question occurs about what will be the stress distribution if the pressure p exceeds the limit load $p_*=2k(1+2\gamma-\pi/2)$ ($p\geq p_*$). The stress–strain state in this problem can be constructed, thus: a stress–strain state governable by (3.4) is realized in domains I and III (see sketch). As in the stationary case, it is assumed that because of the action of a unilateral load the wedge is "bent" then a tensile stress should be expected on the side OD and a compressive stress on the side OA. These two domains are connected by a solution of the form of the double wave obtained in Sec. 1 (domain II).

The arbitrary constants in these solutions are determined from the condition of continuity of the velocity and stress fields on the abutment lines $\varphi=3\pi/2-\gamma_1$ and $\varphi=3\pi/2+\gamma_1$ and from the normal stresses given on the wedge boundaries

$$\begin{gathered}\tau_{r\varphi}=0,\ \sigma_\varphi=-p\ \text{ for }\ \varphi=3\pi/2+\gamma,\\ \tau_{r\varphi}=0,\ \sigma_\varphi=0\ \text{ for }\ \varphi=3\pi/2-\gamma.\end{gathered}$$

The angle γ_1 is also found from these conditions. It is here obtained that first $\zeta=\zeta(p)$ must be found by solving the equation

$$p=(2k-\zeta)(2\gamma_1(\zeta)+\sin 2(\gamma-\gamma_1(\zeta))).\tag{4.1}$$

Here the angle $\gamma_1(\zeta)$ is determined from the relationships $\cos 2(\gamma-\gamma_1(\zeta))=\zeta/(\zeta-2k)$. Afterwards the angle $\gamma_1=\gamma_1(\zeta(p))$ is set up. Consequently, we have the following stress–strain state; the stress field in domain I ($3\pi/2+\gamma_1\leq\varphi\leq 3\pi/2+\gamma$) is

$$2k\cos 2(\theta-\varphi)=\zeta+(2k-\zeta)\cos(3\pi-2\varphi+2\gamma_1),$$

$$\sigma=-p+(2k-\zeta)\sin(\gamma-\varphi+3\pi/2)\cos(\gamma+\varphi-2\gamma_1-3\pi/2)-k\sin 2(\theta-\varphi),$$

the velocity field is

$$v_\varphi=\zeta/r,\ v_r=(-2k\sin 2(\theta-\varphi)+(2k-\zeta)\sin(3\pi+2\gamma_1-2\varphi))/r;$$

in the domain II ($3\pi/2-\gamma_1\leq\varphi\leq 3\pi/2+\gamma_1$)

$$v_r=0,\ v_\varphi=\zeta/r,\ \theta=\varphi-2\pi,$$

$$\sigma=\zeta\varphi-2k\varphi-p+(2k-\zeta)(\sin 2(\gamma-\gamma_1)+3\pi+2\gamma_1)/2;$$

in domain III ($3\pi/2 - \gamma \leq \varphi \leq 3\pi/2 - \gamma_1$) the stress field is

$$2k\cos 2(\theta - \varphi) = \zeta + (2k - \zeta)\cos(3\pi - 2\varphi - 2\gamma_1),$$
$$\sigma = (2k - \zeta)\sin(3\pi/2 - \varphi - \gamma)\cos(3\pi/2 + \gamma - \varphi - 2\gamma_1) - k\sin 2(\theta - \varphi),$$

and the velocity field is

$$v_\varphi = \zeta/r,\ v_r = (-2k\sin 2(\theta - \varphi) + (2k - \zeta)\sin(3\pi - 2\varphi - 2\gamma_1))/r.$$

The stationary solution is realized when $p = p_*$ ($\zeta(p_*) = 0$). Since $(dp/d\zeta)_{\zeta=0} = -8k^2(1 + \gamma - \pi/4)$, then from the theorem about implicit functions the solvability of (4.1) for ζ in the neighborhood of the limit load p_* follows. Here if $p > p_*$, then $\zeta(p) < 0$. When $p \to \infty$, then $\zeta(p) \to -\infty$, while $\gamma_1 \to \gamma$, i.e., the domain II is expanded over the whole wedge.

Remark. The "stationary" part of the velocity (g_1, g_2) is obtained after the construction of the stress field. In particular, if the wedge is in the rest state at the initial time $t = 0$, then the solution will be $g_1 = g_2 = 0$. The line $r = g(\varphi)$ separating the plasticity zone from the rigid rest zone is determined as follows: $g = g_0 = \text{const}$ in domain II while the function $g(\varphi)$ in domain I (III) is found by solving a linear ordinary differential equation $\zeta g' - g(rH_r) = 0$ with the initial conditions $g = g_0$ for $\varphi = 3\pi/2 + \gamma_1$ ($\varphi = 3\pi/2 - \gamma_1$).

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